TENSILE GEOMETRY: A CALCULUS-FREE ARC MEASURE FROM PARAMETRIC CONSTRAINT

JONATHAN CRAIG

Abstract. We introduce the Tensile Length Construct (L_{geo}), a calculus-free geometric measure for cubic Bézier curves derived from parametric constraint rather than integration. Given two orthogonal axes of lengths a and b, a canonical cubic Bézier curve is uniquely determined. The measure L_{geo} is a convex combination of two virtual arc lengths, one at each anchor point, weighted by the axis ratio. We prove L_{geo} is symmetric, monotonically increasing, and saturates at a fixed proportion of the shorter axis as one axis dominates. The supporting framework, Tensile Geometry, provides an analytical toolkit for constructing and analyzing the canonical curve, along with metrics for comparing curves across different axis configurations. The framework's original motivation is the reconstruction of garment patterns from sparse body measurements without calculus-based arc length computation.

1. Introduction

Determining the arc length of a parametric curve conventionally requires evaluating the integral $\int_0^1 \|\dot{B}(t)\|\, dt$, which for cubic Bézier curves has no closed-form solution in general. This paper introduces a calculus-free alternative: the Tensile Length Construct (L_{geo}), a geometric measure derived entirely from the control point structure of a canonical cubic Bézier curve determined by two orthogonal axis lengths.

The construction proceeds from a simple parametric correspondence between orthogonal axes of lengths a and b. This correspondence, formalized as a Tensile Axis Pair, uniquely determines a cubic Bézier curve (the Canonical Curvature Envelope) through the classical control point ratio $k = \frac{4}{3}(\sqrt{2} - 1)$. The curve's curvature is not prescribed by design; it is a geometric consequence of the imposed constraint. We call this approach Tensile Geometry, treating curvature as a "structural residue": the visible trace of parametric tension between the axes.

From this construction, L_{geo} is defined as a weighted sum of two virtual arc lengths at the anchor points. We prove it is a convex combination (the weights partition unity), symmetric, homogeneous of degree one, and monotonically increasing. The measure saturates at ka as one axis dominates, capturing the constraint at the system's bottleneck rather than the curve's total extent. These properties establish L_{geo} as a formally characterized geometric quantity distinct from classical arc length.

A key aspect of this framework is the clarification of three distinct but related uses of the term "tension": (1) the abstract **principle of tensile constraint** that initiates the system; (2) the **Intrinsic Tension Metric**, a local measure of parametric bending stress along the curve; and (3) the **Deviation Energy**, a global L^2 metric that quantifies a curve's departure from its canonical form.

The cubic Bézier approximation to a quarter-circle using the control point ratio $k = \frac{4}{3}(\sqrt{2} - 1)$ is a well-established result in Computer Aided Geometric Design, appearing in foundational work by

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Goldapp [3], Dokken et al. [1], and the standard reference by Farin [2]. The extension to quarter-ellipses via axis-scaled control points is likewise classical. Tensile Geometry does not claim novelty for the control point construction itself, nor for the 0.027% error bound, which is a known property of this approximation. The contributions of this paper are the Tensile Length Construct L_{geo}, a novel geometric measure whose properties are established in Theorem 4.3, the generative interpretation (curvature as a consequence of parametric constraint rather than prescribed by design), and the formal analytical toolkit built around this interpretation (VCP, ACRF, Deviation Energy).

Our primary contributions are:

- The Tensile Length Construct (L_{geo}) and its formal properties: convex combination structure, symmetry, and homogeneity of degree one (Theorem 4.3), monotonicity (Proposition 4.4), and saturation (Proposition 4.5). We conjecture and numerically support a sharp upper bound on the ratio $L_{\text{geo}}/L_{\text{int}}$ (Conjecture 4.6).
- The Tensile Geometry framework: a deterministic construction of canonical cubic Bézier curves from orthogonal axis pairs, with a self-contained proof of the quarter-circle approximation yielding exact maximum radial error $\approx 0.027\%$ (Proposition 3.4, Appendix A).
- An analytical toolkit including the Virtual Circle Pair (VCP), the Virtual Theta with complementarity ($\theta_0 + \theta_1 = \pi/2$), the Anchor–Control Radial Frame (ACRF), and the Strain–Theta Bridge connecting orientational and arc-measure angles through k (Proposition 4.1).
- Deviation Energy (E_{dev}) and Envelope Deviation Energy (E_{env}) as squared L^2 measures for comparing curves and canonical states.

By grounding geometric form in parametric constraint, Tensile Geometry offers a unique generative approach to curvature, opening new avenues for geometric inference, generative design, and the foundational understanding of how simple rules can give rise to complex and structured forms.

2. Preliminaries and Notation

Let the primary coordinate system be $\mathbb{R}^2$. We define two anchor points, A_0 and A_1, which will serve as the endpoints of our canonical curve. For the standard construction in Tensile Geometry, these are placed on orthogonal axes: Let $a, b \in \mathbb{R}^+$ be positive scalar lengths. Define $A_0 = (0, b)$ (on the y-axis) and $A_1 = (a, 0)$ (on the x-axis). The vector connecting these anchor points, oriented from A_0 to A_1, is $\mathbf{u} = A_1 - A_0 = (a, -b)$.

Control points for the Bézier curve will be denoted by P_0 and P_1. The Euclidean inner product is denoted by $\langle \cdot, \cdot \rangle$, and the Euclidean norm by $\| \cdot \|$. All vectors are considered column vectors by default.

The constant k is the canonical Bézier control point ratio, critical for achieving circularity in the symmetric case. It is defined as $k = \frac{4}{3}(\sqrt{2} - 1) \approx 0.5522847498$.

3. Core Tensile Constructs

Definition 3.1 (Tensile Axis Pair and Tensile Field)**.** A *Tensile Axis Pair* is defined by two orthogonal linear extents of lengths a and b, originating from a common origin, say $(0, 0)$. The horizontal axis has length a, and the vertical axis has length b. A parametric correspondence is established: for $t \in [0, 1]$, a point $(a \cdot t, 0)$ on the horizontal axis corresponds to a point $(0, b \cdot (1 - t))$ on the vertical axis. The *Tensile Field*, $\mathcal{F}(a, b, N)$, is the set of N straight line segments connecting these corresponding points for a discrete set of $t_i = i/(N - 1)$ for $i = 0, \dots, N - 1$:

$$\mathcal{F}(a, b, N) = \left\{ \text{Line}\big((a \cdot t_i, 0), (0, b \cdot (1 - t_i))\big) \mid t_i = \frac{i}{N - 1},\ i = 0, \dots, N - 1 \right\}.$$

This field of lines visually suggests an envelope of curvature. The anchor points for the Canonical Curvature Envelope, $A_0 = (0, b)$ and $A_1 = (a, 0)$, are the termini of the lines where $t_i = 0$ and $t_i = 1$ respectively (or vice-versa depending on parameterization direction).

Definition 3.2 (Canonical Curvature Envelope)**.** Given the anchor points $A_0 = (0, b)$ and $A_1 = (a, 0)$, and the canonical control point ratio k, the control points P_0 (associated with A_0) and P_1 (associated with A_1) are defined as:

$$P_0 = (k \cdot a, b) \quad \text{and} \quad P_1 = (a, k \cdot b).$$

The *Canonical Curvature Envelope* is the cubic Bézier curve $B : [0, 1] \to \mathbb{R}^2$ defined by these points:

$$B(t) = (1-t)^3 A_0 + 3(1-t)^2 t P_0 + 3(1-t)t^2 P_1 + t^3 A_1.$$

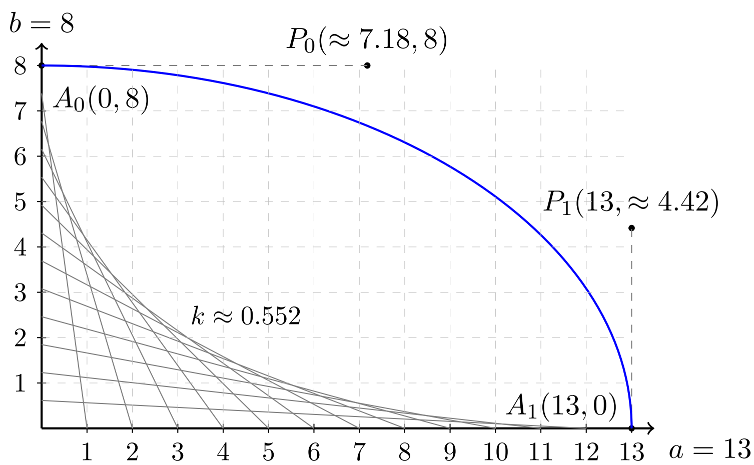

FIGURE 1. Tensile Field with overlaid Canonical Curvature Envelope for $a = 13$, $b = 8$, showing control points and sample tensile field lines (gray). Illustrates the canonical curve induced by axial constraint (Def. 3.2).

Remark 3.3 (On the Term "Envelope")**.** It is important to distinguish the Canonical Curvature Envelope from the strict mathematical envelope of the lines in the Tensile Field (Def. 3.1). The latter is a parabolic curve, whereas the Canonical Curvature Envelope is a cubic Bézier curve. In this framework, the term "Envelope" is used to denote the canonical smooth curve that is *induced* by the field's constraint system, serving as the primary distillation of its geometric properties rather than a curve of literal tangency.

Proposition 3.4 (Quarter-Circle High-Fidelity Approximation)**.** If the tensile axis lengths are equal, i.e., $a = b$, then the Canonical Curvature Envelope $B(t)$ very closely approximates a quarter of a circle of radius a (or b), centered at the origin $(0, 0)$, running from $(0, a)$ to $(a, 0)$, with a maximum radial error of approximately 0.027%.

Proof. When $a = b$, the anchor points are $A_0 = (0, a)$ and $A_1 = (a, 0)$. The control points become $P_0 = (ka, a)$ and $P_1 = (a, ka)$. With $k = \frac{4}{3}(\sqrt{2} - 1)$, these are the standard control points for a cubic Bézier curve to approximate a quarter-circle of radius a connecting $(0, a)$ to $(a, 0)$, with a maximum radial error of approximately 0.027%. A self-contained verification is provided in Appendix A. □

Remark 3.5 (Existence and Uniqueness of the Canonical Envelope)**.** We observe that for any given pair of positive axis lengths $a, b > 0$, and the fixed canonical control point ratio k, there exists a unique cubic Bézier curve serving as the Canonical Curvature Envelope $B(t)$, with endpoints $A_0 = (0, b)$ and $A_1 = (a, 0)$ and control points $P_0 = (ka, b)$ and $P_1 = (a, kb)$. This follows from the constructive nature of the Bézier definition: since a, b, and k are well-defined, the four control points are uniquely determined, and the Bernstein polynomial formula yields exactly one curve.

Definition 3.6 (Virtual Circle Pair (VCP))**.** Associated with the Canonical Curvature Envelope $B(t)$ and its defining points A_0, P_0, P_1, A_1: Let $r_0 = \|P_0 - A_0\|$ be the distance from the first anchor point to its control point. Let $r_1 = \|P_1 - A_1\|$ be the distance from the second anchor point to its control point. The ordered pair (r_0, r_1) is the *Virtual Circle Pair*. For the canonical envelope: $r_0 = \|(ka, b) - (0, b)\| = \|(ka, 0)\| = ka$. $r_1 = \|(a, kb) - (a, 0)\| = \|(0, kb)\| = kb$.

Definition 3.7 (Virtual Theta (θ_V))**.** The *Virtual Theta* measures the rotational skew of a control point's influence relative to the primary axis of the curve. It is the angle between the anchor-to-anchor chord vector and the anchor-to-control-point vector.

Let $\mathbf{u} = A_1 - A_0 = (a, -b)$ be the vector along the chord. Let $\mathbf{v}_0 = P_0 - A_0 = (ka, 0)$ be the vector from the first anchor to its control point. Let $\mathbf{v}_1 = P_1 - A_1 = (0, kb)$ be the vector from the second anchor to its control point.

The Virtual Theta angles are defined as:

$$\theta_0 = \arccos\left(\frac{\langle \mathbf{u}, \mathbf{v}_0\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}_0\|}\right)$$
$$\theta_1 = \arccos\left(\frac{\langle -\mathbf{u}, \mathbf{v}_1\rangle}{\|-\mathbf{u}\|\,\|\mathbf{v}_1\|}\right)$$

The pair (θ_0, θ_1) is the *Virtual Theta*, a general-purpose measure of orientational constraint used in constructs such as the Anchor-Control Radial Frame.

Remark 3.8 (Virtual Theta for Canonical Quarter-Circle)**.** For the canonical quarter-circle where $a = b$, we can calculate the exact value of the Virtual Theta angles. In this case, $\mathbf{u} = (a, -a)$, $\|\mathbf{u}\| = a\sqrt{2}$, $\mathbf{v}_0 = (ka, 0)$, and $\mathbf{v}_1 = (0, ka)$. The dot products are $\langle \mathbf{u}, \mathbf{v}_0\rangle = ka^2$ and $\langle -\mathbf{u}, \mathbf{v}_1\rangle = ka^2$. The magnitudes are $\|\mathbf{v}_0\| = ka$ and $\|\mathbf{v}_1\| = ka$. This yields:

$$\theta_0 = \theta_1 = \arccos\left(\frac{ka^2}{a\sqrt{2}\cdot ka}\right) = \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

Thus, for the symmetric quarter-circle, the rotational skew at both anchors is exactly 45 degrees, as expected for a canonical circular arc segment.

Remark 3.9 (Complementarity of Virtual Theta)**.** For any tensile axis pair (a, b) with $a, b > 0$, the Virtual Theta angles satisfy $\theta_0 + \theta_1 = \frac{\pi}{2}$. This follows directly: since $\theta_0 = \arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right)$ and $\frac{a}{\sqrt{a^2+b^2}} = \sin(\theta_1)$, we have $\theta_0 = \arccos(\sin(\theta_1)) = \frac{\pi}{2} - \theta_1$. The two ACRF frames thus partition the chord angle into complementary halves.

Definition 3.10 (Anchor–Control Radial Frame (ACRF))**.** For each anchor point A_i ($i \in \{0, 1\}$), the *ACRF* at A_i is the local polar coordinate frame defined by:

- **Origin**: The anchor point A_i.
- **Reference Direction**: The vector $\mathbf{u} = A_1 - A_0$ for the frame at A_0, and $-\mathbf{u} = A_0 - A_1$ for the frame at A_1.
- **Radial Coordinate**: The radius r_i from the Virtual Circle Pair (Def. 3.6).
- **Angular Coordinate**: The angle θ_i from the Virtual Theta (Def. 3.7).

The ACRF thus describes the position of the control point P_i in polar terms (r_i, θ_i) relative to A_i and the curve's chord.

4. Arc Length of the Canonical Curvature Envelope

Two methods for determining the arc length of the Canonical Curvature Envelope $B(t)$ are considered.

4.1. Integral Arc Length (L_{int}). The classical arc length of the Bézier curve $B(t)$ from $t = 0$ to $t = 1$ is given by the integral of the magnitude of its first derivative:

$$L_{\text{int}}(a, b) = \int_0^1 \|\dot{B}(t)\| dt,$$

where $\dot{B}(t) = \frac{dB}{dt}(t)$. The derivative $\dot{B}(t)$ is:

$$\dot{B}(t) = 3(1-t)^2(P_0 - A_0) + 6(1-t)t(P_1 - P_0) + 3t^2(A_1 - P_1).$$

This integral generally does not have a closed-form solution for arbitrary a and b and is typically evaluated numerically (e.g., using Gaussian quadrature).

In the special case where $a = b$, the Canonical Curvature Envelope closely approximates a quarter-circle of radius a, whose exact arc length is $\pi a/2$. The arc length of the Bézier approximation itself is

$$L_{\text{int}}(a, a) \approx \frac{\pi a}{2},$$

with the two values differing by less than 0.015% (the same order as the radial approximation error).

4.2. Tensile Length Construct (L_{geo}). In contrast to the classical integral method, we introduce the *Tensile Length Construct*, a geometric approach to quantifying the curve's properties without calculus. This construct is not intended to replicate the integral arc length, but rather to serve as a proxy for the tensile arc measure of the system: a measure of how much "bending" was required to resolve the imposed tensile constraint.

The calculation of this tensile arc measure requires a specific type of angle that measures the control point's displacement relative to the global coordinate axes, which is distinct from the Virtual Theta (Def. 3.7) used for rotational analysis. We therefore define a set of *Strain Angles*, ϕ_0 and ϕ_1:

$$\phi_0 = \arctan\left(\frac{ka}{b}\right)$$
$$\phi_1 = \arctan\left(\frac{kb}{a}\right)$$

Proposition 4.1 (Strain–Theta Bridge)**.** The Strain Angles are determined by the Virtual Theta and the canonical constant k alone:

$$\phi_0 = \arctan(k \cot\theta_0), \qquad \phi_1 = \arctan(k \tan\theta_0).$$

Proof. Since $\theta_0 = \arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right)$, we have $\tan(\theta_0) = b/a$ and $\cot(\theta_0) = a/b$. Substituting into the definitions $\phi_0 = \arctan(ka/b)$ and $\phi_1 = \arctan(kb/a)$ gives the result. □

This identity establishes a structural bridge between the ACRF angular system and the Tensile Length Construct: the canonical constant k mediates the transformation from orientational analysis (θ_i) to arc measure (ϕ_i). The Strain Angles carry no independent geometric information beyond the Virtual Theta and k.

Using these angles, along with the radii $r_0 = ka$, $r_1 = kb$ from the VCP (Def. 3.6) and participation weights α_0, α_1, the Tensile Length Construct is defined. The weights are given by:

$$\alpha_0 = \frac{2}{\pi}\arctan\left(\frac{b}{a}\right), \quad \alpha_1 = \frac{2}{\pi}\arctan\left(\frac{a}{b}\right).$$

The construct is then:

$$L_{\text{geo}}(a,b) = \alpha_0 r_0 \phi_0 + \alpha_1 r_1 \phi_1.$$

This value reframes the curve's length-like properties not as something to be extracted by calculus, but as a quantity demanded by the system's relational structure.

The Tensile Length Construct serves a distinct purpose from the integral arc length L_{int}. Where L_{int} measures total distance traveled along the curve, L_{geo} measures the constraint intensity imposed by the axis pair at the anchor points. Its closed-form algebraic structure admits exact partial derivatives with respect to a and b, enabling gradient-based optimization over curve families without numerical differentiation. Its convex combination structure decomposes the measure into individually meaningful per-anchor contributions, identifying which axis drives the curve's constraint. Its saturation at k times the shorter axis bounds the measure even as one axis grows without limit, tracking the geometric bottleneck rather than the curve's total extent. These properties make L_{geo} applicable in any setting where paired orthogonal parameters jointly determine a curve, and the operative question is not "how long is the curve" but "how much constraint does the parameter pair impose."

Remark 4.2 (Interpretation of L_{geo})**.** For the symmetric case where $a = b$, we can evaluate L_{geo} using the Strain Angles: $r_0 = ka$, $r_1 = ka$. The Strain Angles become $\phi_0 = \arctan(k)$ and $\phi_1 = \arctan(k)$. The weights become $\alpha_0 = \frac{2}{\pi}\arctan(1) = \frac{1}{2}$ and $\alpha_1 = \frac{1}{2}$.

Substituting these values into the formula:

$$L_{\text{geo}}(a,a) = \left(\frac{1}{2}\right)(ka)\arctan(k) + \left(\frac{1}{2}\right)(ka)\arctan(k) = ka\arctan(k).$$

With $k \approx 0.55228$, $L_{\text{geo}}(a,a) \approx 0.55228 \cdot a \cdot 0.505 \approx 0.2789a$. This value represents the canonical tensile arc measure of the symmetric system, a baseline against which asymmetric configurations can be compared.

Theorem 4.3 (Properties of the Tensile Length Construct)**.** *For all $a, b > 0$, the Tensile Length Construct L_{geo} satisfies:*

(i) **Convex combination.** The participation weights form a partition of unity: $\alpha_0 + \alpha_1 = 1$. Consequently, $L_{\text{geo}}(a,b)$ is a convex combination of two virtual arc lengths $r_0\phi_0$ and $r_1\phi_1$.

(ii) **Symmetry.** $L_{\text{geo}}(a,b) = L_{\text{geo}}(b,a)$.

(iii) **Homogeneity.** $L_{\text{geo}}(\lambda a, \lambda b) = \lambda L_{\text{geo}}(a,b)$ for all $\lambda > 0$. The construct is homogeneous of degree one.

Proof. **(i)** By definition, $\alpha_0 + \alpha_1 = \frac{2}{\pi}\arctan\left(\frac{b}{a}\right) + \frac{2}{\pi}\arctan\left(\frac{a}{b}\right) = \frac{2}{\pi}\left[\arctan\left(\frac{b}{a}\right) + \arctan\left(\frac{a}{b}\right)\right]$. For $x > 0$, the identity $\arctan(x) + \arctan(1/x) = \pi/2$ holds. Setting $x = b/a$ yields $\alpha_0 + \alpha_1 = \frac{2}{\pi} \cdot \frac{\pi}{2} = 1$. Since $\alpha_0, \alpha_1 \in (0,1)$ and $\alpha_0 + \alpha_1 = 1$, the expression $L_{\text{geo}} = \alpha_0(r_0\phi_0) + \alpha_1(r_1\phi_1)$ is a convex combination.

(ii) Under the substitution $(a,b) \mapsto (b,a)$: $r_0 = ka \mapsto kb$, $r_1 = kb \mapsto ka$; $\phi_0 = \arctan(ka/b) \mapsto \arctan(kb/a)$, $\phi_1 = \arctan(kb/a) \mapsto \arctan(ka/b)$; $\alpha_0 = \frac{2}{\pi}\arctan(b/a) \mapsto \frac{2}{\pi}\arctan(a/b)$, $\alpha_1 = \frac{2}{\pi}\arctan(a/b) \mapsto \frac{2}{\pi}\arctan(b/a)$. The two terms exchange roles: the value of $\alpha_0 r_0 \phi_0 + \alpha_1 r_1 \phi_1$ is invariant.

(iii) Under $(a,b) \mapsto (\lambda a, \lambda b)$: $r_0 = ka \mapsto k\lambda a = \lambda r_0$ and similarly $r_1 \mapsto \lambda r_1$. The Strain Angles $\phi_i = \arctan(k \cdot (\text{ratio of } a, b))$ depend only on a/b, so they are unchanged. The weights α_i likewise depend only on a/b. Thus $L_{\text{geo}}(\lambda a, \lambda b) = \alpha_0(\lambda r_0)\phi_0 + \alpha_1(\lambda r_1)\phi_1 = \lambda L_{\text{geo}}(a,b)$. □

Proposition 4.4 (Monotonicity)**.** L_{geo} is monotonically increasing in each argument separately: for fixed $b > 0$, $\partial L_{\text{geo}}/\partial a > 0$ for all $a > 0$, and symmetrically for b.

Proof (analytic structure with numerical verification). By homogeneity (Theorem 4.3(iii)), $\partial L_{\text{geo}}/\partial a$ depends only on the ratio $\rho = a/b$. Setting $b = 1$ and writing $f(\rho) = L_{\text{geo}}(\rho, 1)$, the sign of $\partial L_{\text{geo}}/\partial a$ equals the sign of $f'(\rho)$. Symbolic differentiation yields an expression whose denominator is a product of π, (ρ^2+1), $(9\rho^2 + 16(1-\sqrt{2})^2)$, and $(16\rho^2(1-\sqrt{2})^2 + 9)$, all strictly positive for $\rho > 0$. The numerator is a sum of terms involving $\arctan(1/\rho)$, $\arctan(k\rho)$, and $\arctan(\rho)$, each multiplied by positive polynomial factors. At the boundary, $\lim_{\rho\to 0^+} f'(\rho) = k \approx 0.5523$ and $\lim_{\rho\to\infty} f'(\rho) = 0^+$, with $f'(1) \approx 0.1393$. Numerical evaluation on a logarithmically spaced grid of 10,000 points over $\rho \in [10^{-4}, 10^4]$ confirms $f'(\rho) > 0$ everywhere, with minimum value $\approx 3.3 \times 10^{-9}$ (attained as $\rho \to \infty$). The argument for b follows by symmetry (Theorem 4.3(ii)). □

Proposition 4.5 (Saturation Limit)**.** For fixed $a > 0$,

$$\lim_{b\to\infty} L_{\text{geo}}(a,b) = ka.$$

As one axis dominates, the Tensile Length Construct saturates at k times the shorter axis.

Proof. As $b \to \infty$: $\alpha_0 = \frac{2}{\pi}\arctan(b/a) \to 1$ and $\alpha_1 \to 0$. For the first term: $r_0\phi_0 = ka \cdot \arctan(ka/b) \to ka \cdot 0 = 0$. For the second term: $\alpha_1 \cdot r_1\phi_1 = \frac{2}{\pi}\arctan(a/b) \cdot kb \cdot \arctan(kb/a)$. Using $\arctan(x) \to \pi/2$ as $x \to \infty$ and $\arctan(1/x) \sim 1/x$ for large x: $\alpha_1 \cdot r_1\phi_1 \sim \frac{2}{\pi} \cdot \frac{a}{b} \cdot kb \cdot \frac{\pi}{2} = ka$. Thus $L_{\text{geo}}(a,b) \to 0 + ka = ka$. □

Conjecture 4.6 (Sharp Bound on $L_{\text{geo}}/L_{\text{int}}$)**.** For all $a, b > 0$,

$$\frac{L_{\text{geo}}(a,b)}{L_{\text{int}}(a,b)} \le \frac{2k\arctan(k)}{\pi} \approx 0.1774,$$

with the maximum attained at $a = b$.

By Theorem 4.3(iii), both L_{geo} and L_{int} are homogeneous of degree one, so their ratio depends only on the axis ratio $r = b/a$. Define $f(r) = L_{\text{geo}}(1,r)/L_{\text{int}}(1,r)$. Numerical computation over a dense grid ($r \in [0.01, 100]$, 10,000 sample points) confirms f is unimodal with a unique maximum at $r = 1$, satisfies $f(r) = f(1/r)$, and decays to zero as $r \to 0^+$ or $r \to \infty$. The first derivative vanishes at $r = 1$ and the second derivative is negative (≈ -0.092), confirming a strict local maximum.

The stated bound uses $L_{\text{int}}(a,a) = \pi a/2$ (exact quarter-circle arc length) in the symmetric case. Since the Bézier curve's arc length differs from the true quarter-circle by less than 0.015%, the bound's numerical value is stable to four significant figures under either convention.

b/a	$L_{\text{geo}}(1,b/a)$	$L_{\text{int}}(1,b/a)$	$L_{\text{geo}}/L_{\text{int}}$
1	0.2787	1.5710	0.1774
2	0.3498	2.2386	0.1563
5	0.3973	5.0671	0.0784
10	0.4078	10.0334	0.0406

5. Deviation Energy and Curvature Regimes

5.1. **Deviation Energy (E_{dev}).** To quantify how much an arbitrary curve $C(t)$ (sharing the same endpoints A_0, A_1 and parameterization $t \in [0,1]$) deviates from the Canonical Curvature Envelope $B(t)$ associated with specific a, b values, we define the *Deviation Energy*:

$$E_{\text{dev}}(C, B) = \int_0^1 \|C(t) - B(t)\|^2 dt.$$

This is the squared L^2 distance between the two curves.

Proposition 5.1. $E_{\text{dev}}(C, B) = 0$ if and only if $C(t) = B(t)$ for all $t \in [0,1]$.

This follows directly from the positive-definiteness of the L^2 norm on the space of continuous functions on $[0,1]$.

The Deviation Energy can be used as an objective function in optimization problems. A related concept is a "tension metric field" over a morph space of curves, where $B(t)$ represents the zero-energy state for its defining (a, b) pair.

5.2. **Envelope Deviation Energy (E_{env}).** To measure the difference between two distinct Canonical Curvature Envelopes, $B_r(t)$ (generated by axis ratio $r = b/a$) and $B_{r'}(t)$ (generated by $r' = b'/a'$), we define the *Envelope Deviation Energy*:

$$E_{\text{env}}(B_r, B_{r'}) = \int_0^1 \|B_r(t) - B_{r'}(t)\|^2 dt.$$

This provides a measure of dissimilarity on the space of canonical envelopes, quantifying the squared L^2 distance between different canonical states corresponding to different axis ratios.

Remark 5.2 (Example Calculation of E_{env})**.** We can calculate the energy cost required to deform the canonical quarter-circle into a simple asymmetric state. Let $B_1(t)$ be the canonical envelope for the symmetric case ($a = 1, b = 1$), and let $B_2(t)$ be the envelope for an asymmetric case where the horizontal axis is unchanged but the vertical axis is halved ($a = 1, b = 0.5$).

The x-component of the Bézier curve depends only on a, which is unchanged, so $x_1(t) = x_2(t)$. The y-component is scaled by b, so $y_2(t) = 0.5 \cdot y_1(t)$. The difference vector is $B_1(t) - B_2(t) = (0, 0.5 \cdot y_1(t))$.

The Envelope Deviation Energy is therefore:

$$E_{\text{env}}(B_1, B_2) = \int_0^1 \|(0, 0.5 \cdot y_1(t))\|^2 dt = 0.25 \int_0^1 (y_1(t))^2 dt.$$

For the canonical quarter-circle approximation with radius $a = 1$, numerical integration yields $\int_0^1 (y_1(t))^2 dt \approx 0.50014$ (a very close approximation to $1/2$ with error $\approx 0.028\%$). The energy cost is thus:

$$E_{\text{env}}(B_1, B_2) \approx 0.25 \cdot 0.50014 = 0.125035.$$

This value represents the specific energetic cost of halving the vertical tensile axis relative to the symmetric equilibrium state.

5.3. **Curvature Regimes.** The morphology of the Canonical Curvature Envelope $B(t)$ varies significantly with the ratio of the tensile axis lengths, $r_{\text{axis}} = b/a$. We can classify these morphologies into *Curvature Regimes*. Let $\rho(r_{\text{axis}}) = \log_2(\max\{r_{\text{axis}}, 1/r_{\text{axis}}\})$ be a regime score measuring deviation from symmetry ($r_{\text{axis}} = 1$).

The following classifications are descriptive, based on qualitative morphological behavior rather than formally derived transition thresholds.

- **Circular Equilibrium ($\rho = 0$, i.e., $r_{\mathbf{axis}} = 1 \implies a = b$):** The envelope very closely approximates a perfect quarter-circle. Curvature is symmetric.
- **Balanced Asymmetry ($0 < \rho < 1$, e.g., $0.5 < r_{\mathbf{axis}} < 2$ but $r_{\mathbf{axis}} \neq 1$):** The envelope is a smooth, stable arc, skewed towards the dominant axis.
- **Spiral Skew ($\rho \geq 1$, e.g., $r_{\mathbf{axis}} \leq 0.5$ or $r_{\mathbf{axis}} \geq 2$):** The envelope concentrates curvature sharply near the dominant axis. The curve remains convex for all $a, b > 0$; the term "Spiral Skew" describes the visual impression of tightly concentrated curvature, not literal spiraling.
- **Critical Transition Zones (e.g., $\rho \approx 1$):** Regions where the morphology transitions more rapidly.

This classification aids in understanding the qualitative behavior of tensile curvature.

6. Intrinsic Tension Metric along the Canonical Envelope

Distinct from energy of deviation *from* a canonical curve, the intrinsic "bending stress" or local curvature intensity *along* the Canonical Curvature Envelope $B(t)$ itself is also a key property.

Definition 6.1 (Intrinsic Tension Metric)**.** The *Intrinsic Tension Metric*, $\mathcal{T}_B(t)$, along the Canonical Curvature Envelope $B(t)$ is defined as the magnitude of its second derivative with respect to the parameter t:

$$\mathcal{T}_B(t) = \|\ddot{B}(t)\| = \left\| \frac{d^2B}{dt^2}(t) \right\|.$$

The second derivative $\ddot{B}(t)$ for a cubic Bézier curve with anchor points A_0, A_1 and control points P_0, P_1 (where P_0 is the control point associated with A_0, and P_1 with A_1) is:

$$\ddot{B}(t) = 6(1-t)(P_1 - 2P_0 + A_0) + 6t(A_1 - 2P_1 + P_0).$$

This scalar field $\mathcal{T}_B(t)$ quantifies the local intensity of curvature at each point t along the canonical path. We note $\mathcal{T}_B(t)$ measures the magnitude of parametric acceleration, which is related to but distinct from the geometric curvature $\kappa(t) = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}}$. The two coincide (up to scaling) only for arc-length-parameterized curves.

Remark 6.2 (Intrinsic Tension of the Quarter-Circle)**.** For the symmetric quarter-circle case where $a = b$, we can calculate the Intrinsic Tension Metric at key points along the curve. The value is not constant. Using the formula for $\ddot{B}(t)$ above, we can find the tension at the endpoints ($t = 0$ and $t = 1$) and at the midpoint ($t = 1/2$).

At the endpoints, the tension is:

$$\mathcal{T}_B(0) = \mathcal{T}_B(1) = 6a\sqrt{5k^2 - 6k + 2} \approx 2.759a.$$

At the midpoint of the curve, the tension is at its minimum:

$$\mathcal{T}_B(1/2) = 3\sqrt{2}ak = a(8 - 4\sqrt{2}) \approx 2.343a.$$

This demonstrates that for the canonical quarter-circle, the parametric "bending stress" is highest at the anchor points where the curve meets the axes, and is most relaxed at the center of the arc.

7. Discussion and Conclusion

The central contribution of this work is the Tensile Length Construct (L_{geo}), a calculus-free measure whose formal properties (Theorem 4.3) and saturation behavior (Proposition 4.5) establish it as a characterized geometric quantity. The construct is a convex combination of virtual arc lengths at the two anchor points, symmetric under axis exchange, homogeneous of degree one, and bounded above relative to the integral arc length (Conjecture 4.6). The supporting framework, Tensile

Geometry, furnishes an analytical toolkit connecting orientational analysis (Virtual Theta, ACRF) to arc measure (Strain Angles) through the canonical constant k (Proposition 4.1).

Beyond its geometric origins, L_{geo}'s algebraic properties function as general-purpose primitives for any system characterized by paired competing parameters. The convex combination structure provides smooth, bounded, unity-summing attribution between two causes: the participation weights α_i allocate the measure between anchors without reaching the extremes of zero or one, reflecting that neither parameter is ever fully irrelevant. The regime score $\rho = \log_2(\max\{b/a, a/b\})$ classifies parameter balance on a logarithmic scale invariant to absolute magnitude. The saturation limit identifies which of two competing parameters is the binding constraint. These properties transfer to domains beyond computational geometry, including resource allocation under paired demands, adaptive systems responding to imbalanced state, and any design context where a scalar measure of paired-parameter tension is needed with closed-form computability, exact differentiability, and structural decomposability.

7.1. **Future Directions.** The generative nature of Tensile Geometry opens numerous avenues for theoretical and applied research. The most immediate direction is the development of a canonical decomposition: algorithms and theory to express arbitrary curves as a unique combination or sequence of Canonical Curvature Envelopes, establishing the envelopes as a true basis set for describing geometric form. A second direction is the formal development of sizeless pattern systems, using Tensile Geometry to reconstruct 2D garment patterns from sparse 3D human body measurements, which was the original motivation for this work. Finally, the framework invites investigation of recursive structures and surfaces: applying tensile principles recursively to generate complex, multi-segment curves or fractal-like geometries, and extending the 2D framework to the generation and analysis of 3D Tensile Surfaces.

7.2. **Conclusion.** This paper has introduced the Tensile Length Construct, a calculus-free geometric measure whose properties are formally established and whose structure as a convex combination of virtual arc lengths is proved. The supporting framework, Tensile Geometry, provides a deterministic path from orthogonal axis constraints to canonical Bézier curvature, equipped with an integrated analytical toolkit. By demonstrating that complex, structured curves can arise deterministically from simple, abstract rules, this work offers a new lens through which we can explore the nature of form itself, reminding us that sometimes the most intricate shapes are born not from complex design, but from relational necessity.

Appendix A. Auxiliary Proofs and Derivations

A.1. **Proof of Proposition 3.4 (Quarter-Circle High-Fidelity Approximation).** The proposition states that for the symmetric case where the tensile axis lengths are equal ($a = b$), the Canonical Curvature Envelope is a high-fidelity approximation of a quarter-circle of radius a.

Proof. To prove this, we must show that for all $t \in [0,1]$, the coordinates of the curve $B(t) = (x(t), y(t))$ closely approximate the equation of a circle, $x(t)^2 + y(t)^2 \approx a^2$, with a maximum radial error of approximately 0.027%.

1. Define the control polygon points for the symmetric case. When $a = b$, the anchor and control points are:

$$A_0 = (0, a)$$
$$P_0 = (ka, a)$$
$$P_1 = (a, ka)$$
$$A_1 = (a, 0)$$

Where the canonical control point ratio k is given by $k = \frac{4}{3}(\sqrt{2} - 1)$.

2. Write the Bézier curve equation. The cubic Bézier curve is defined as $B(t) = (1-t)^3 A_0 + 3(1-t)^2 t P_0 + 3(1-t)t^2 P_1 + t^3 A_1$. We separate this into its x and y components.

3. Express the x-coordinate, $x(t)$**.**

$$\begin{aligned} x(t) &= (1-t)^3(0) + 3(1-t)^2 t(ka) + 3(1-t)t^2(a) + t^3(a) \\ &= a\left[3kt(1-t)^2 + 3t^2(1-t) + t^3\right] \\ &= a\left[3kt(1-2t+t^2) + 3t^2 - 3t^3 + t^3\right] \\ &= a\left[3kt - 6kt^2 + 3kt^3 + 3t^2 - 2t^3\right] \\ &= a\left[3kt + (3-6k)t^2 + (3k-2)t^3\right] \end{aligned}$$

4. Express the y-coordinate, $y(t)$**.**

$$\begin{aligned} y(t) &= (1-t)^3(a) + 3(1-t)^2 t(a) + 3(1-t)t^2(ka) + t^3(0) \\ &= a\left[(1-t)^3 + 3t(1-t)^2 + 3kt^2(1-t)\right] \\ &= a\left[(1-3t+3t^2-t^3) + 3t(1-2t+t^2) + 3kt^2 - 3kt^3\right] \\ &= a\left[1 - 3t + 3t^2 - t^3 + 3t - 6t^2 + 3t^3 + 3kt^2 - 3kt^3\right] \\ &= a\left[1 - 3t^2 + 2t^3 + 3kt^2 - 3kt^3\right] \\ &= a\left[1 + (3k-3)t^2 + (2-3k)t^3\right] \end{aligned}$$

5. Verify the circle equation. Define the dimensionless residual $R(t)/a^2 = (x(t)/a)^2 + (y(t)/a)^2 - 1$. Substituting the expressions from Steps 3 and 4 and expanding yields a degree-6 polynomial in t. Differentiating and solving for critical points gives $t = 0$, $t = \frac{1}{2}$, $t = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$, and $t = 1$. The residual vanishes exactly at $t = 0$, $t = \frac{1}{2}$, and $t = 1$. The maximum of $|R(t)/a^2|$ occurs at $t = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$, where

$$\left|\frac{R(t)}{a^2}\right| = \frac{17}{54} - \frac{2\sqrt{2}}{9} \approx 0.0005451.$$

Converting to radial error, $\max_t \left|\frac{\sqrt{x(t)^2+y(t)^2}}{a} - 1\right| = \frac{\sqrt{426-72\sqrt{2}}}{18} - 1 \approx 0.0002725$, which is approximately 0.027%.

Thus, the curve very closely approximates a quarter-circle of radius a from $(0, a)$ to $(a, 0)$. □

Appendix B. Computational Details

This section summarizes the final formulas for computation, using the standard anchor points $A_0 = (0, b)$ and $A_1 = (a, 0)$, and control points $P_0 = (ka, b)$ and $P_1 = (a, kb)$.

B.1. **ACRF Parameters and Strain Angles.** Tensile Geometry uses two distinct types of angles for analysis.

Virtual Theta (θ_i): Used for general rotational analysis within the ACRF.

$$r_0 = ka, \quad r_1 = kb$$
$$\mathbf{u} = (a, -b), \quad \|\mathbf{u}\| = \sqrt{a^2 + b^2}$$
$$\theta_0 = \arccos\left(\frac{a}{\sqrt{a^2 + b^2}}\right)$$
$$\theta_1 = \arccos\left(\frac{b}{\sqrt{a^2 + b^2}}\right)$$

Note: These angles are well-defined if $\|\mathbf{u}\| \neq 0$.

Strain Angles (ϕ_i): Used specifically for calculating the Tensile Length Construct (L_{geo}).

$$\phi_0 = \arctan(ka/b) \quad \text{(handle } b = 0 \text{ case)}$$
$$\phi_1 = \arctan(kb/a) \quad \text{(handle } a = 0 \text{ case)}$$

B.2. **Tensile Length Construct (L_{geo}).** The formula for L_{geo} uses the radii (r_i), Strain Angles (ϕ_i), and participation weights (α_i).

$$\alpha_0 = \frac{2}{\pi}\arctan(b/a) \quad \text{(handle } a = 0 \text{ case)}$$
$$\alpha_1 = \frac{2}{\pi}\arctan(a/b) \quad \text{(handle } b = 0 \text{ case)}$$
$$L_{\text{geo}} = \alpha_0 r_0 \phi_0 + \alpha_1 r_1 \phi_1$$

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